Hays Lyle, an alumnus of Westminster College, a Theological student of Princeton Seminary for two years, and at present a minister in charge of a church at La Junta, Colorado.

Dr. Lyle, in 1884, married his second wife, Miss Mattie E. Grant, a scholarly and cultured lady, of Bardstown, Kentucky.

Dr. Lyle has been for many years an Elder in the Presbyterian Church, the church of his ancestors for, at least, the century and a half that have elapsed since his Great Grandfather emigrated from the northern part of Ireland to Berkeley County, Virginia.

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

[Concluded.]

We will now find the centroid of the eighth part of the surface

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1, \text{ I, when } c=b, \text{ II, } c=a.$$

We have $\bar{x}=\dfrac{\int xds}{\int ds}, \quad \bar{y}=\dfrac{\int yds}{\int ds}, \quad \bar{z}=\dfrac{\int zds}{\int ds}.$

$$\text{I.} \quad s=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}dxdy$$

$$=\frac{\pi b}{2a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx=\tfrac{1}{4}\pi b(b+\frac{a}{e}\sin^{-1}e).$$

$$s.\bar{x}=\int xds=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}xdxdy$$

$$= \frac{\pi b}{2a^2}\int_o^a \sqrt{a^4-(a^2-b^2)x^2}\, xdx = \frac{\pi ab(a^2+ab+b^2)}{6(a+b)}$$

$$\therefore \bar{x} = \frac{2a(a^2+ab+b^2)}{3(a+b)(b+\frac{a}{e}\sin^{-1}e)}.$$

$$s.\bar{y} = s.\bar{z} = \int yds = \frac{b}{a}\int_o^a\int_o^{\frac{b}{a}\sqrt{a^2-x^2}} \left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}} ydxdy$$

$$= \frac{b^2}{a^2}\int_o^a \sqrt{(a^2-x^2)(a^2-e^2x^2)}\,dx = ab^2\int_o^{\frac{1}{2}\pi}\sqrt{1-e^2\sin^2\theta}\cos^2\theta d\theta,\ x=a\sin\theta$$

$$= \frac{ab^2}{3e^2}\left\{(1+e^2)\,E(e,\frac{\pi}{2})-(1-e^2)F(e,\frac{\pi}{2})\right\}.$$

$$\therefore \bar{y} = \bar{z} = \frac{4ab\left\{(1+e^2)E(e,\frac{\pi}{2})-(1-e^2)F(e,\frac{\pi}{2})\right\}}{3\pi e^2(b+\frac{a}{e}\sin^{-1}e)}.$$

II. $$s = \frac{a}{b}\int_o^b\int_o^{\frac{a}{b}\sqrt{b^2-y^2}}\left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}} dydx$$

$$= \frac{\pi a}{2b^2}\int_o^b \sqrt{b^4+(a^2-b^2)y^2}\,dy = \frac{\pi a^2}{4}\left\{1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right\}.$$

$$s.\bar{x} = s.\bar{z} = \int xds = \frac{a}{b}\int_o^b\int_o^{\frac{a}{b}\sqrt{b^2-y^2}}\left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}} xdydx.$$

$$s.\bar{x} = s.\bar{z} = \frac{a^2}{b^3}\int_o^b \sqrt{(b^2-y^2)(b^4+a^2e^2y^2)}\,dy$$

$$= a^2\int_o^{\frac{1}{2}\pi}\sqrt{b^2+a^2e^2\cos^2\theta}\sin^2\theta d\theta,\ y = b\cos\theta$$

$$= a^3\int_o^{\frac{1}{2}\pi}\sqrt{1-e^2\sin^2\theta}\sin^2\theta d\theta$$

$$=\frac{a^3}{3e^2}\left\{(1-e^2)F(e,\frac{\pi}{2})-(1-2e^2)E(e,\frac{\pi}{2})\right\}.$$

$$\therefore\ \bar{x}=\bar{z}=\frac{4a\left\{(1-e^2)F(e,\frac{\pi}{2})-(1-2e^2)E(e,\frac{\pi}{2})\right\}}{3\pi e^2(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e})}.$$

$$s.\bar{y}=\frac{a}{b}\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}}\left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}ydydx=\int yds$$

$$=\frac{\pi a}{2b^2}\int_0^b\sqrt{b^4+a^2e^2y^2}\,ydy=\frac{\pi ab(a^2+ab+b^2)}{6(a+b)}.$$

$$\therefore\ \bar{y}=\frac{2b(a^2+ab+b^2)}{3a(a+b)(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e})}.$$

Since the limit of $\frac{\sin^{-1}e}{e}$ and $\frac{\log\frac{1+e}{1-e}}{2e}$ is 1 when $e=0$ we have, in either case, when $a=b$, $\bar{x}=\bar{y}=\bar{z}=\frac{1}{2}a$. The surface of the fourth part of the paraboloid $x^2+y^2=2a^2z$, for $z=h$.

$$s=\int\int\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dz}\right)^2}\,dzdx=\int\int\sqrt{1+\frac{x^2}{y^2}+\frac{a^4}{y^2}}\,dxdz.$$

$$\therefore\ s=a\int_0^h\int_0^{a\sqrt{2z}}\sqrt{\frac{a^2+2z}{2a^2z-x^2}}\,dzdx=\frac{\pi a}{2}\int_0^h\sqrt{a^2+2z}\,dz$$

$$=\frac{\pi a}{6}\left\{(a^2+2h)^{\frac{3}{2}}-a^3\right\}.$$

$$s.\bar{x}=s.\bar{y}=\int yds=a\int_0^h\int_0^{a\sqrt{2z}}\sqrt{a^2+2z}\,dzdx=a^2\int_0^h\sqrt{(a^2+2z)2z}\,dz$$

$$=\frac{a^2}{16}\left\{2(a^2+4h)\sqrt{2a^2h+4h^2}-a^4\log\left(\frac{a^2+4h+\sqrt{2a^2h+4h^2}}{a^2}\right)\right\}.$$

$$\therefore\ \bar{x}=\bar{y}=\frac{3a\left\{2(a^2+4h)\sqrt{2a^2h+4h^2}-a^4\log\left(\frac{a^2+4h+2\sqrt{2a^2h+4h^2}}{a^2}\right)\right\}}{8\pi\{(a^2+2h)^{\frac{3}{2}}-a^3\}}.$$

$$s.\bar{z}=\int zds=a\int_o^h\int_o^{a\sqrt{2z}}\sqrt{\frac{a^2+2z}{2a^2z-x^2}}zdzbx$$

$$=\frac{\pi a}{2}\int_o^h\sqrt{a^2+2z}\,zdz=\frac{\pi a}{30}\left\{(3h-a^2)(a^2+2h)^{\frac{3}{2}}+a^5\right\}.$$

$$\therefore\bar{z}=\frac{(3h-a^2)(a^2+2h)^{\frac{3}{2}}+a^5}{5\{(a^2+2h)^{\frac{3}{2}}-a^3\}}.$$

The surface of the fourth part of the cone $x^2+y^2=a^2z^2$, for $z=h$.

$$s=\iint\sqrt{1+\frac{x^2}{y^2}+\frac{a^4z^2}{y^2}}\,dzdx=a\sqrt{1+a^2}\int_o^h\int_o^{az}\frac{zdzdx}{\sqrt{a^2z^2-x^2}}$$

$$=\frac{\pi a\sqrt{1+a^2}}{2}\int_o^h zdz=\frac{\pi ah^2\sqrt{1+a^2}}{4}.$$

$$s.\bar{x}=s.\bar{y}=\int yds=a\sqrt{1+a^2}\int_o^h\int_o^{az}zdzdx=a^2\sqrt{1+a^2}\int_o^h z^2dx=\frac{a^2h^3\sqrt{1+a^2}}{3}.$$

$$\therefore\ \bar{x}=\bar{y}=\frac{4ah}{3\pi}.$$

$$s.\bar{z}=\int zds=a\sqrt{1+a^2}\int_o^h\int_o^{az}\frac{z^2dzdx}{\sqrt{a^2z^2-x^2}}=\frac{\pi a\sqrt{1+a^2}}{2}\int_o^h z^2dz=\frac{\pi ah^3\sqrt{1+a^2}}{6}.$$

$$\therefore\ \bar{z}=\frac{2h}{3}.$$